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Spin glass stiffness in zero dimensions

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Abstract

A unique analytical result for the Migdal–Kadanoff hierarchical lattice is obtained. The scaling of the defect energy for a zero-dimensional spin glass is derived for a bond distribution that is continuous at the origin. The value of the ‘stiffness’ exponent in zero dimensions, $y_0 = -1$, corresponds to the value also found in one dimension. This result complements and completes earlier findings for y_d at $d > 0$.

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A quantity of fundamental importance for the modelling of amorphous magnetic materials through spin glasses [1] is the ‘stiffness’ exponent y [2, 3]. The stiffness of a spin configuration describes the typical rise in magnetic energy ΔE due to an induced defect interface of size L . In a glassy system, the potential energy function resembles a high-dimensional mountain landscape over its variables [4]. Any defect-induced displacement of size L in such a landscape may move a system equally likely up or down in energy ΔE . Averaging over many incarnations of such a system then results in a typical energy scale

$$\langle |\Delta E| \rangle \sim L^y \quad (L \rightarrow \infty). \quad (1)$$

The importance of this exponent for small excitations in disordered spin systems has been discussed in many contexts [1–3, 5–8]. In particular, it signifies a renormalized coupling strength (across any hypothetical interface) between regions in space separated by a distance L [3]: if $y_d > 0$, regions in space are strongly coupled at low temperature and spin glass ordering ensues, i.e. $T_g > 0$.

Reference [9] provided a description of y_d as a continuous function of dimension d using a fit to the data obtained in [10] for $d = 2, 3, \dots, 6$. That fit became credible in that it reproduced the exactly known result in $d = 1$, $y_1 = -1$, to within less than 1%. Hence, it validated the values for y_d found in [10, 11] and produced a number of predictions such as that $d_l = 5/2$ may be the *lower critical dimension* (the dimension in which $y_d = 0$) for Ising spin glasses, in accordance with an earlier calculation invoking replica symmetry breaking [12].

In a quest for understanding universality in spin glasses, there has been considerable interest recently in the behaviour of y_d even for $d < d_l$, where any spin glass ordering is unstable. Presumably, for divergent energy scales [8] in equation (1), i.e. for $y_d > 0$,

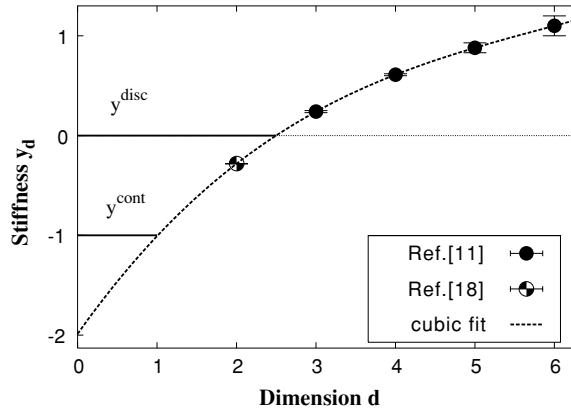


Figure 1. Plot of the spin glass stiffness exponent y_d as a function of dimension d . Shown are the values for y_d on hyper-cubic lattices from [11, 18] and a cubic fit to that data. For bounded distributions, the exponent remains locked at $y^{\text{disc}} \equiv 0$ for $d \leq d_l = 5/2$ (upper horizontal line). For continuous distributions, the fit reproduces the exact result, $y_1 = -1-0.8\%$ and suggests $y_0 = -2$. In contrast, the result here suggests $y_0^{\text{cont}} = -1$ and, hence, $y^{\text{cont}} \equiv -1$ for all $d \leq 1$ (lower horizontal line).

universality holds and low-temperature properties of the system are independent of the details of the bond distribution, as long as it possesses a zero mean and unit variance. In contrast, below the lower critical dimension significant differences have been found between classes of bond distributions [8, 13, 14]. With $y < 0$, energy scales contract under renormalization, which magnifies details of the bond distribution near $J = 0$. Especially, for all $d < d_l$, a discrete bond distribution ($J = \pm 1$) (which has an energy gap near $J = 0$) leads to trivial scaling in equation (1), as was found numerically for $d = 2$ [13] and is exactly known for $d = 1$. (In a linear chain of L spins, the $T = 0$ energy difference $|\Delta E|$ for reversed boundary conditions is given by the smallest bond weight, $|J| = 1$, independent of L .) Only bond distributions $P(J)$ which are continuous near the origin $P(0)$ obtain non-trivial scaling as represented by the curve in [9], including the exact result $y_1 = -1$. (Here, the smallest bond weight $|J|$ in the chain approaches zero with $1/L$.)

In this paper, we report on a (rather fortuitous) analytical result for a zero-dimensional spin glass that further clarifies the behaviour of $y_{d < d_l}$ for a continuous bond distribution. In the Migdal–Kadanoff hierarchical lattice (MK) [2, 3, 8, 14] we find for a $d = 0$ dimensional spin glass that $y_0 = -1$ exactly.

To our knowledge, aside from $d = 1$ (where MK is trivially exact) and the large-link limit [8], this is the only exact result for MK applied to spin glasses. While not of great practical relevance, studying physical systems in unphysical dimensions has proved to be of significant theoretical relevance [15, 16]. A zero-dimensional spin glass in particular has in fact been considered previously in [17].

Our result suggests a behaviour for y_d as depicted in figure 1: while for a discrete bond distribution it is $y_d^{\text{disc}} \equiv 0$ for all $d \leq d_l$, for a continuous bond distribution, y_d first extends smoothly to (non-trivial) negative values through d_l towards $d = 1$, beyond which it appears to get fixed at $y_d^{\text{cont}} \equiv -1$ for all $d \leq 1$. As long as $y_d > 0$ for $d > d_l$ the exponent is believed to be universal, independent of the bond distribution.

The Migdal–Kadanoff (MK) hierarchical lattice [19, 20] provides a real-space renormalization scheme that approximates especially low-dimensional spin glasses well and

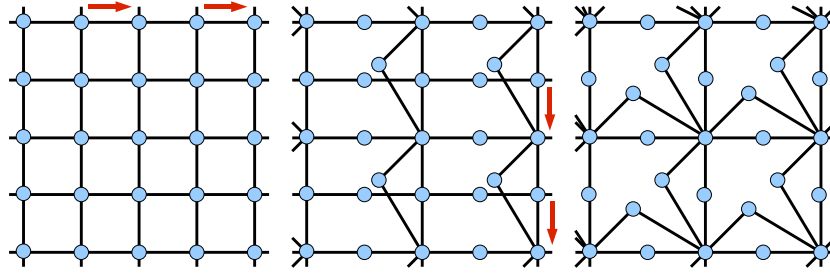


Figure 2. Bond-moving scheme in the Migdal–Kadanoff hierarchical lattice, here for a square lattice ($d = 2$) with $l = 2$, i.e. $b = 2$ in equation (2). Starting from the lattice with unit bonds (left), bonds in intervening hyper-planes are projected onto every l th plane in one direction (middle), then subsequent directions, to re-obtain a similar hyper-cubic lattice of bond-length l (right). The renormalized bonds in this case consist of $b = 2$ branches, each of a series of $l = 2$ bonds.

(This figure is in colour only in the electronic version)

is trivially exact in $d = 1$. These lattices have a simple recursive, yet geometric, structure and are well studied [2, 3, 8, 21]. Starting from generation I with a single bond, at each subsequent generation $I + 1$, all bonds from I are replaced with a new sub-graph. This structure of the sub-graph arises from the bond-moving scheme, as shown in figure 2, in d dimensions [19, 20]: in a hyper-cubic lattice of unit bond length, at first all $l - 1$ intervening hyper-planes of bonds, transverse to a chosen direction, are projected onto every l th hyper-plane, followed by the same step for $l - 1$ hyper-planes being projected onto the l th plane in the next direction and so on. In the end, one obtains a renormalized hyper-cubic lattice (of bond length l) in generation $I + 1$ with a reformulated $(I + 1)$ -bond consisting of a sub-graph of

$$b = l^{d-1} \quad (2)$$

parallel branches of a series of lI -bonds each. We can rewrite equation (2) as

$$d = 1 + \frac{\ln(b)}{\ln(l)}, \quad (3)$$

anticipating analytic continuation in l and b to obtain results in arbitrary dimensions d .

Instead of solving the problem on the hyper-cubic lattice, we merely need to consider the recursive scheme of obtaining the bond distribution in generation $I + 1$ from sub-graphs of bonds from generation I . Numerically, this is done efficiently at any temperature T to yield a stationary bond distribution for $I \rightarrow \infty$, i.e. the thermodynamic limit $L = l^I \rightarrow \infty$.

Here, we are only concerned with $T = 0$, which simplifies the calculation drastically to the point that analytical results can be obtained. For instance, a series of l bonds can always be replaced by the bond of smallest absolute weight. Thus, if these bonds are drawn from a distribution $P_I(J)$, then the distribution $Q_I^{(l)}(K)$ of the effective bond K replacing the series can be obtained from

$$\begin{aligned} Q_I^{(l)}(K) &= \mathcal{P}\{K = \text{sign}(J_1 \times J_2 \times \dots \times J_l) \min(|J_1|, \dots, |J_l|)\}, \\ &\propto \int_{-\infty}^{\infty} dJ_l P_I(J_l) \int_{-\infty}^{\infty} dJ_{l-1} P_I(J_{l-1}) \Theta(|J_l| - |J_{l-1}|) \\ &\quad \dots \int_{-\infty}^{\infty} dJ_1 P_I(J_1) \Theta(|J_2| - |J_1|) \delta(|J_1| - |K|), \end{aligned} \quad (4)$$

where $\Theta(x)$ refers to the unit step function and $\delta(x) = \Theta'(x)$ is the Dirac delta-function. In writing equation (4), we have dropped the obvious norm of $Q^{(l)}(K)$. We have also exploited

the symmetry of the integrand under relabelling J_1, \dots, J_l which in principle consists of a sum of $l!$ such terms, each for a different ordering in the magnitude of $|J_i|$.

Similarly, having b bonds drawn from a distribution $Q^{(l)}(K)$ in parallel leads to the distribution $P_{l+1}(J)$ of the reformulated bond at generation $l + 1$,

$$\begin{aligned} P_{l+1}(J) &= \int_{-\infty}^{\infty} \prod_{i=1}^b [dK_i Q^{(l)}(K_i)] \delta\left(J - \sum_{i=1}^b K_i\right), \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega J} \left[\int_{-\infty}^{\infty} dK Q^{(l)}(K) e^{-i\omega K} \right]^b, \end{aligned} \quad (5)$$

where we have used the integral representation $\delta(x) = \int_{-\infty}^{\infty} d\omega e^{i\omega x} / (2\pi)$.

For general P_l , equations (4) and (5) both are quite complicated. This is particularly true, since we would be interested in finding the limiting shape of $P_l(J)$ after infinitely many iterations, $l \rightarrow \infty$. In that limit, we expect the effective coupling between domains a distance $L = l^l$ apart to scale as [3]

$$\mathcal{J}_{\text{eff}}(L) \sim l^y \mathcal{J}_{\text{eff}}(L/l). \quad (6)$$

Each iteration of the characteristic ‘width’ $\mathcal{J}_{\text{eff}}(L) \equiv [\langle J^{2n} \rangle_l]^{1/2n} \sim \langle |\Delta E| \rangle_L$ (any $n = 1, 2, \dots$) of $P_l(J)$ increases (or decreases) by a factor of l^y , so

$$P_{l+1}(J) \sim \frac{1}{l^y} P_l\left(\frac{J}{l^y}\right) \quad (l \rightarrow \infty). \quad (7)$$

Clearly, if the width grew smaller, $l^y < 1$, the behaviour of $P_l(J)$ near $J = 0$ would become increasingly relevant, explaining the non-universal behaviour there [8].

We can now report on a solution for equations (4) and (5) for *all* values of l for a specific set of initial distributions and choices of l and b , appropriately analytically continued. With those choices, only one iteration of equations (4) and (5) is necessary, since the distribution remains *shape invariant*, i.e. equation (7) becomes an equality. This set of distributions is continuous and finite near the origin and thus should represent the universality class containing Gaussian bonds, for instance. Starting more generally with an initial distribution

$$P_0(J) = \frac{q+1}{2J_0} \left(1 - \frac{|J|}{J_0}\right)^q \Theta\left(1 - \frac{|J|}{J_0}\right) \quad (q > -1), \quad (8)$$

where $J_0 > 0$ sets the energy scale for this distribution, it is easy to show that

$$Q_0^{(l)}(K) = \frac{l(q+1)}{2J_0} \left(1 - \frac{|K|}{J_0}\right)^{lq+l-1} \Theta\left(1 - \frac{|K|}{J_0}\right), \quad (9)$$

by recursion of equation (4). Note that equation (9) readily continues to any real value of $l > 0$.

The evaluation of equation (5) is somewhat more complex. To facilitate the subsequent analysis, it is best to consider the moment generating function for $P_l(J)$,

$$\begin{aligned} \phi_{l+1}(\alpha) &= \langle e^{-i\alpha J} \rangle_{l+1}, \\ &= \int_{-\infty}^{\infty} dJ e^{-i\alpha J} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega J} \left[\int_{-\infty}^{\infty} dK Q^{(l)}(K) e^{-i\omega K} \right]^b, \\ &= \left[\int_{-\infty}^{\infty} dK Q^{(l)}(K) e^{-i\alpha K} \right]^b, \end{aligned} \quad (10)$$

where the integral over J merely represents a δ -function. Using equation (9), we find

$$\begin{aligned}\phi_1(\alpha) &= \left[\int_{-\infty}^{\infty} dK Q_0^{(l)}(K) e^{-i\alpha K} \right]^b, \\ &= \left[l(q+1) \int_0^1 dx (1-x)^{lq+l-1} \cos(\alpha J_0 x) \right]^b.\end{aligned}\quad (11)$$

Correspondingly, we find for the generating function of the initial bond distribution in equation (8)

$$\phi_0(\alpha) = \langle e^{-i\alpha J} \rangle_0 = (q+1) \int_0^1 dx (1-x)^q \cos(\alpha J_0 x). \quad (12)$$

Despite the obvious similarities between equations (11) and (12), finding a set of parameters that make ϕ_0 and ϕ_1 similar is hard because of the exponent b in equation (11). Even for $b = 1$ and $l \neq 1$, i.e. $d = 1$ according to equation (3), we need to iterate for $l \rightarrow \infty$ to find the width to decay to zero with $1/L$, i.e. $y_1 = -1$. In our efforts, we have found merely two solutions, both pertaining to zero dimensions, with the required properties, each independently yielding the same result.

First, for $l = 2$ we have to continue the ‘branching number’ to $b = 1/2$ to obtain $d = 0$ in equation (3). Then, for the rectangular function for $P_0(J)$, i.e. $q = 0$, we find

$$\phi_0(\alpha) = \frac{\sin(\alpha J_0)}{\alpha J_0}, \quad \phi_1(\alpha) = \left[\frac{4 \sin^2\left(\frac{\alpha J_0}{2}\right)}{(\alpha J_0)^2} \right]^{\frac{1}{2}}, \quad (13)$$

both of which are invariant in a sufficiently large open interval around $\alpha = 0$, required to generate any moment. We identify in equation (6) $\mathcal{J}_{\text{eff}}(L = l^0) = J_0$ and $\mathcal{J}_{\text{eff}}(L = l^1) = J_0/2 = l^y \mathcal{J}_{\text{eff}}(L = l^0)$, hence, $y_0 = -1$.

Second, for $b = 2$ we have to continue the ‘series number’ to $l = 1/2$ to obtain $d = 0$ in equation (3). Then, for the triangular function for $P_0(J)$, i.e. $q = 1$, we find

$$\phi_0(\alpha) = \frac{4 \sin^2\left(\frac{\alpha J_0}{2}\right)}{(\alpha J_0)^2}, \quad \phi_1(\alpha) = \frac{\sin^2(\alpha J_0)}{(\alpha J_0)^2}. \quad (14)$$

In this case, we identify in equation (6) again $\mathcal{J}_{\text{eff}}(L = l^0) = J_0$, but $\mathcal{J}_{\text{eff}}(L = l^1) = 2J_0$, which now results again in $y_0 = -1$ because length scales are actually shrinking, $l = 1/2$.

It is clear that nothing will change on this result under further iteration, $l \rightarrow l + 1$. Assuming that there is a unique solution for equation (7), one would expect that any choice for $P_0(J)$ that is continuous at $J = 0$ should converge to a rectangular (triangular) function for $l = 1/b = 2$ ($= \frac{1}{2}$). We have not been able to extract any further result of this nature.

In conclusion, we are led to believe that $y_0 = -1$. Of course, such a result has to be considered with care. It is a weakness of MK that its results for a given dimension d according to equation (3) are not generally unique [9]. Here, at least, we found two different combinations of l and b , both giving $d = 0$ and yielding an identical stiffness exponent. This may indicate a unique result for any combination $l = 1/b$. Furthermore, it is not clear that the MK result for $d = 0$ should necessarily correspond *quantitatively* to an Edwards–Anderson spin glass on a zero-dimensional lattice. This is only known to be true for $d = 1$ (i.e., $b = 1$) and certainly wrong for increasing $d > 1$, albeit with slowly increasing error.

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